# Chapter 1

# Kähler manifolds

## 1.1 Manifolds

Let $M$ be an $2n$-dimensional manifold. Let $TM$ be the tangent vector bundle over $M$. Let $\mathrm{End}(TM)$ be the real vector bundle over $M$ such that the fibre $\mathrm{End}(TM)|_x$ for any $x \in M$ is canonically isomorphic to $\mathrm{End}(TM|_x)$. Let $E$ be a vector bundle over $M$. Let $\mathscr{C}^\infty(M, E)$ be the space of smooth sections of $E$ on $M$. Let $\Omega^r(M, E)$ be the smooth $r$-forms on $M$ with values in $E$.

**Definition 1.1.1.** The manifold $M$ is called a **almost complex manifold** if there exists $J \in \mathscr{C}^\infty(M, \mathrm{End}(TM))$ such that $J^2 = -\mathrm{Id}$. The endomorphism $J$ is called the **almost complex structure** of $TM$.

For $x \in M$, the almost complex structure $J$ induces a splitting of complex vector spaces,

$$T_xM \otimes_{\mathbb{R}} \mathbb{C} = T_x^{(1,0)}M \oplus T_x^{(0,1)}M, \tag{1.1.1}$$

where $T_x^{(1,0)}M$ and $T_x^{(0,1)}M$ are the eigenspaces of $J$ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Since $J$ is smooth, $T^{(1,0)}M = \{T_x^{(1,0)}M\}_{x\in M}$ and $T^{(0,1)}M = \{T_x^{(0,1)}M\}_{x\in M}$ are vector bundles.

A continuous map $\pi : E \to M$ between two Hausdorff spaces is called a complex vector bundle of rank $r$ if for any $x \in M$, $E_x := \pi^{-1}(x)$ is a complex vector space of dimension $r$ and there is a neighbourhood $U$ of $x$ and a homeomorphism

$$\psi : \pi^{-1}(U) \to U \times \mathbb{C}^r \tag{1.1.2}$$

such that for any $p \in U$, $\psi(E_p) = \{p\} \times \mathbb{C}^r$ and $\psi|_{E_p}$ is a complex linear space isomorphism. The pair $(U, \psi)$ is called a local trivialization. For a complex vector bundle $\pi : E \to M$, $E$ is called the total space and $M$ the base space. We often say that $E$ is a vector bundle over $M$. Notice that for two local trivializations $(U_i, \psi_i)$ and $(U_j, \psi_j)$, the map $\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^r \to (U_i \cap U_j) \times \mathbb{C}^r$ induces a transition map

$$\psi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C}). \tag{1.1.3}$$

When $r = 1$, we will call $E$ a complex line bundle.

It is easy to see that The eigenbundles $T^{(1,0)}M$ and $T^{(0,1)}M$ are complex vector bundles over $M$.

Let $T^{*(1,0)}M$ and $T^{*(0,1)}M$ be the dual bundles respectively. We denote by

$$\Omega^{p,q}(M) := \mathscr{C}^\infty(M, \Lambda^p(T^{*(1,0)}M) \otimes \Lambda^q(T^{*(0,1)}M)). \tag{1.1.4}$$

By (1.1.1), we have

$$\Omega^k(M, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(M). \tag{1.1.5}$$

If $\alpha \in \Omega^{p,q}(M)$, we say that $\alpha$ is a $(p,q)$-form. For $\alpha \in \Omega^{p,q}(M)$, from (1.1.5), we have $d\alpha = \sum_{j+k=p+q+1} (d\alpha)^{(j,k)}$, where $(d\alpha)^{(j,k)} \in \Omega^{j,k}(M)$. We define

$$\partial\alpha = (d\alpha)^{(p+1,q)}, \quad \bar\partial\alpha = (d\alpha)^{(p,q+1)}. \tag{1.1.6}$$

Let $g$ be any Riemannian metric on $TM$ compatible with $J$, i.e.,

$$g(JU, JV) = g(U, V) \tag{1.1.7}$$

for any $U, V \in T_xM$, $x \in M$.

Take $e_1 \in T_xM$. Then $g(Je_1, e_1) = 0$ by $J^2 = -\mathrm{Id}$. Take orthonormal vectors $e_1, \cdots, e_k \in T_xM$. If $e_{k+1} \notin \mathrm{span}\{e_1, Je_1, \cdots e_k, Je_k\}$ , then so is $Je_{k+1}$. So we can construct an orthonormal basis of $T_xM$ with the form $\{e_1, \cdots, e_{2n}\}$ such that $e_{n+i} = Je_i$, $1 \le i \le n$. Moreover,

$$\begin{aligned} T_x^{(1,0)}M &= \mathbb{C}\{e_1 - \sqrt{-1}e_{n+1}, \cdots, e_n - \sqrt{-1}e_{2n}\}, \\ T_x^{(0,1)}M &= \mathbb{C}\{e_1 + \sqrt{-1}e_{n+1}, \cdots, e_n + \sqrt{-1}e_{2n}\}. \end{aligned} \tag{1.1.8}$$

We also denote by $g$ the $\mathbb{C}$-bilinear form on $TM \otimes_{\mathbb{R}} \mathbb{C}$ induced by $g$ on $TM$. From (1.1.8), we could see that $g$ vanishes on $T^{(1,0)}M \times T^{(1,0)}M$ and $T^{(0,1)}M \times T^{(0,1)}M$. That is, for any $Z, Z' \in T^{(1,0)}M$,

$$g(Z, Z') = g(\bar Z, \overline{Z'}) = 0. \tag{1.1.9}$$

Let

$$\theta_j = \frac{1}{\sqrt{2}}(e_j - \sqrt{-1}e_{n+j}), \quad \bar{\theta}_j = \frac{1}{\sqrt{2}}(e_j + \sqrt{-1}e_{n+j}), \quad 1 \le j \le n. \tag{1.1.10}$$

Then $\{\theta_j\}_{1\le j\le n}$ and $\{\bar{\theta}_j\}_{1\le j\le n}$ form orthonormal basis of complex vector spaces $T_x^{(1,0)}M$ and $T_x^{(0,1)}M$ respectively. Let $\{\theta^j\}_{1\le j\le n}$ be the dual frame of $\{\theta_j\}_{1\le j\le n}$. Let $e^i$ be the dual of $e_i$. We have

$$\theta^j = \frac{1}{\sqrt{2}}(e^j + \sqrt{-1}e^{n+j}), \quad \bar{\theta}^j = \frac{1}{\sqrt{2}}(e^j - \sqrt{-1}e^{n+j}), \quad 1 \le j \le n. \tag{1.1.11}$$

From (1.1.11), we have

$$(\sqrt{-1})^n \theta^1 \wedge \cdots \wedge \theta^n \wedge \bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n = e^1 \wedge \cdots \wedge e^{2n}. \tag{1.1.12}$$

**Proposition 1.1.2.** *The almost complex manifold is orientable.*

*Proof.* Let $\{e'_1, \cdots, e'_{2n}\}$ be another basis of $TM$. We may assume that $e'_{n+i} = Je'_i$, $1 \le i \le n$. Then there exists $\alpha \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ such that $\theta'^1 \wedge \cdots \wedge \theta'^n = \alpha \cdot \theta^1 \wedge \cdots \wedge \theta^n$. Since $\bar{\theta}'^1 \wedge \cdots \wedge \bar{\theta}'^n = \bar{\alpha} \cdot \bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n$, from (1.1.12), we have $e'^1 \wedge \cdots \wedge e'^{2n} = |\alpha|^2 e^1 \wedge \cdots \wedge e^{2n}$.

So our proposition follows from $|\alpha|^2 > 0$. □

Let $\omega$ be the real 2-form defined by

$$\omega(X,Y) = g(JX,Y) \tag{1.1.13}$$

for vector fields $X, Y$. Note that $\omega(X,Y) = -\omega(Y,X)$ follows from $g(JX,Y) = g(J^2X, JY) = -g(JY,X)$. Since $g$ is non-degenerate, by (1.1.9), $\omega$ is a non-degenerate real $(1,1)$-form. Since $g$ is compatible with $J$, so is $\omega$.

Conversely, we have the following lemma.

**Lemma 1.1.3.** *If there exists a non-degenerate real 2-form $\omega$ on $M$, then $M$ is almost complex.*

*Proof.* Choose a metric $g$ on $TM$. Since $\omega$ is real and non-degenerate, there exists invertible skew-symmetric $A \in \mathscr{C}^\infty(M, \mathrm{End}(TM))$ such that

$$\omega(X,Y) = g(AX,Y) \tag{1.1.14}$$

for any vector fields $X, Y$. Since $-A^2$ is positive definite, $(-A^2)^{1/2}$ is invertible. Then our lemma follows by defining $J = ((-A^2)^{1/2})^{-1}A$. □

Note that fixing a non-degenerate real $(1,1)$-form $\omega$ on almost complex manifold $(M, J)$ compatible with $J$, we could construct a Riemannian metric compatible with $J$ by

$$g(X,Y) = \omega(X, JY) \tag{1.1.15}$$

for vector fields $X$ and $Y$.

**Definition 1.1.4.** A triple $(g, J, \omega)$ satisfying (1.1.7) and (1.1.13) is called a **compatible triple** of almost complex manifold $M$.

**Definition 1.1.5.** A **complex manifold** is a manifold with an atlas of charts to the open unit disk in $\mathbb{C}^n$, such that the transition maps are holomorphic.

**Proposition 1.1.6.** *The complex manifold is almost complex.*

*Proof.* Let $\{z^1, \cdots z^n\}$ be a local chart of a complex manifold $M$ with complex dimension $n$. Denote by $z^k = x^k + \sqrt{-1}y^k$. Then $\{x^1, y^1, \cdots, x^n, y^n\}$ is a local chart of $M$ as real manifold. So $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\}$ is a real basis of $TM$. For $x \in M$, the linear transform $J_x : T_xM \to T_xM$ is defined by

$$J_x\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}, \quad J_x\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}. \tag{1.1.16}$$

Obviously, $J_x^2 = -\operatorname{Id}$.

We claim that the definition of $J_x$ does not depend on the coordinates. In fact, let $\{\theta^1, \cdots, \theta^n\}$ be another local complex chart of $M$. Then by the definition of complex manifold, $z^j$ is holomorphic on $\theta^k$ for any $1 \le j, k \le n$. That is, for $\theta^k = u^k + \sqrt{-1}v^k$, we have $\frac{\partial x^j}{\partial u^k} = \frac{\partial y^j}{\partial v^k}$, $\frac{\partial x^j}{\partial v^k} = -\frac{\partial y^j}{\partial u^k}$ (Cauchy-Riemann equation). Therefore,

$$\begin{aligned} J_x\left(\frac{\partial}{\partial u^k}\right) &= J_x\left(\frac{\partial x^j}{\partial u^k}\frac{\partial}{\partial x^j} + \frac{\partial y^j}{\partial u^k}\frac{\partial}{\partial y^j}\right) = \frac{\partial}{\partial v^k}, \\ J_x\left(\frac{\partial}{\partial v^k}\right) &= J_x\left(\frac{\partial x^j}{\partial v^k}\frac{\partial}{\partial x^j} + \frac{\partial y^j}{\partial v^k}\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial u^k}. \end{aligned}$$

So the endomorphism $J$ in (1.1.16) is global defined.

The proof of our proposition is completed. □

For a complex manifold $M$, the almost complex structure defined in (1.1.16) is called the canonical almost complex structure of $M$. Moreover,

$\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\}$ and $\{\frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n}\}$ are basis of $T_x^{(1,0)}M$ and $T_x^{(0,1)}M$ respectively. Let $dz^i$, $d\bar{z}^i$ be the duals of $\frac{\partial}{\partial z^i}$, $\frac{\partial}{\partial \bar{z}^i}$ respectively. Then

$$dz^i = dx^i + \sqrt{-1}dy^i, \quad d\bar{z}^i = dx^i - \sqrt{-1}dy^i \tag{1.1.17}$$

and

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right). \tag{1.1.18}$$

From (1.1.9), locally for any $1 \le i, j \le n$,

$$g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right) = 0. \tag{1.1.19}$$

We write

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right), \quad g_{\bar{i}j} = g\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}\right). \tag{1.1.20}$$

Then

$$g_{i\bar{j}} = \overline{g_{\bar{i}j}}. \tag{1.1.21}$$

From (1.1.13) and (1.1.18), we have

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j. \tag{1.1.22}$$

We could easily check that the right hand side of (1.1.22) does not depend on the basis.

The Nijenhuis tensor $N^J : TM \times TM \to TM$ is given by

$$N^J(V, W) = [V, W] + J[JV, W] + J[V, JW] - [JV, JW] \tag{1.1.23}$$

for $V$, $W$ vector fields on $M$.

**Theorem 1.1.7** (Newlander-Nirenberg)**.** *Let $(M, J)$ be a almost complex manifold. The following statements are equivalent:*

*(1) $M$ is a complex manifold and $J$ is the canonical almost complex structure of $M$.*

*(2) $T^{(1,0)}M$ is formally integrable, that is, for any $X, Y \in \mathscr{C}^\infty(M, T^{(1,0)}M)$, $[X, Y] \in \mathscr{C}^\infty(M, T^{(1,0)}M)$.*

*(3) $T^{(0,1)}M$ is formally integrable.*

*(4) $N^J = 0$.*

*(5) On $\Omega^{(1,0)}(M)$, $d = \partial + \bar{\partial}$.*

*(6) On $\Omega^{(p,q)}(M)$, $d = \partial + \bar{\partial}$.*

*(7) $\bar{\partial}^2 = 0$.*

*If $(M, J)$ satisfies one of the above statements, we say that the almost complex structure $J$ is integrable.*

*Proof.* (1)$\Rightarrow$(2): Write $X = X^i \frac{\partial}{\partial z^i}$ and $Y = Y^j \frac{\partial}{\partial z^j}$. Then

$$[X,Y] = X^i \frac{\partial Y^j}{\partial z^i}\frac{\partial}{\partial z^j} + Y^j \frac{\partial X^i}{\partial z^j}\frac{\partial}{\partial z^i} \in \mathscr{C}^\infty(M, T^{(1,0)}M).$$

(2)$\Leftrightarrow$(3) follows from $\overline{[X,Y]} = [\overline{X}, \overline{Y}]$.

(3)$\Leftrightarrow$(4): For $X, Y \in \mathscr{C}^\infty(M, TM)$, then $X + \sqrt{-1}JX$, $Y + \sqrt{-1}JY \in \mathscr{C}^\infty(M, T^{(0,1)}M)$. Let $Z = [X + \sqrt{-1}JX, Y + \sqrt{-1}JY]$. It is easy to calculate that $Z - \sqrt{-1}JZ = N^J(X,Y) - \sqrt{-1}JN^J(X,Y)$. So $Z \in \mathscr{C}^\infty(M, T^{(0,1)}M) \Leftrightarrow N^J(X,Y) = 0$.

(3)$\Leftrightarrow$(5): (5) is equivalent to that for any $\theta \in \Omega^{(1,0)}(M)$, $(d\theta)^{(0,2)} = 0$. For $X, Y \in \mathscr{C}^\infty(M, T^{(0,1)}M)$,

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]) = -\theta([X,Y]).$$

So

$$\begin{aligned} d\theta(X,Y) = 0 \quad \forall \theta \in \Omega^{(1,0)}(M), X, Y \in \mathscr{C}^\infty(M, T^{(0,1)}M) \\ \Leftrightarrow \theta([X,Y]) = 0 \quad \forall \theta \in \Omega^{(1,0)}(M), X, Y \in \mathscr{C}^\infty(M, T^{(0,1)}M) \\ \Leftrightarrow [X,Y] \in \mathscr{C}^\infty(M, T^{(0,1)}M) \quad \forall X, Y \in \mathscr{C}^\infty(M, T^{(0,1)}M) \end{aligned}$$

(5)$\Leftrightarrow$(6): Suppose (5) holds. By complex conjugation, on $\Omega^{(0,1)}(M)$, we have $d = \partial + \bar\partial$. Then (6) follows from the Leibniz rule.

(6)$\Rightarrow$(7) follows from $d^2 = 0$.

(7)$\Rightarrow$(5): Let $\{\theta^1, \cdots, \theta^n\}$ be a local frame of $T^{*(1,0)}M$. Let

$$d\theta^i = A^i_{jk}\theta^j \wedge \theta^k + B^i_{jk}\theta^j \wedge \bar\theta^k + C^i_{jk}\bar\theta^j \wedge \bar\theta^k.$$

Then (4) is equivalent to $C^i_{jk} = 0$, $\forall 1 \le i, j, k \le n$. Let $f : M \to \mathbb{C}$ be a smooth function. Then

$$\begin{aligned} 0 = \bar\partial^2 f = (d(\bar\partial f))^{(0,2)} = (d((\bar\partial - d)f))^{(0,2)} = -(d(\partial f))^{(0,2)} \\ = -\theta_i(f) C^i_{jk}\bar\theta^j \wedge \bar\theta^k. \end{aligned}$$

Since $f$ is chosen arbitrarily, $C^i_{jk} = 0$, $\forall 1 \le i, j, k \le n$.

We will not prove (2)$\Rightarrow$(1) here. This part is very hard. We leave a reference to the reader. □

The only spheres which admit almost complex structures are $S^2$ and $S^6$ (Borel-Serre, 1953). In particular, $S^4$ cannot be given an almost complex structure (Ehresmann and Hopf). Whether the $S^6$ has a complex structure is an open question.

**Definition 1.1.8.** Let $\omega$ be a non-degenerate real valued 2-form on $M$. If $d\omega = 0$, $\omega$ is called a symplectic form on $M$. In this case, $(M, \omega)$ is called a **symplectic manifold**.

The following Proposition follows from Lemma 1.1.3.

**Proposition 1.1.9.** *The symplectic manifold is almost complex.*

**Definition 1.1.10.** Let $(g, J, \omega)$ be the compatible triple on almost complex manifold $M$ defined in Definition 1.1.4. If one of the statements in Theorem 1.1.7 holds and $d\omega = 0$, $M$ is called a **Kähler manifold**. In this case, $\omega$ is called the **Kähler form** and $g$ is called the **Kähler metric**.

**Example 1.1.11.** Let $M = \mathbb{C}^n$. Then from (1.1.22),

$$\omega = \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i = \sum_{i=1}^{n} dx_i \wedge dy_i. \tag{1.1.24}$$

is a Kähler form of $\mathbb{C}^n$.

**Example 1.1.12 (Projective space).** The complex projective space $\mathbb{CP}^n$ is the set of complex lines in $\mathbb{C}^{n+1}$ or, equivalently,

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^*, \tag{1.1.25}$$

where $\mathbb{C}^*$ acts by multiplication on $\mathbb{C}^{n+1}$. The topology of $\mathbb{CP}^n$ is induced by (1.1.25). The points of $\mathbb{CP}^n$ are written as $[z_0 : z_1 : \cdots : z_n]$ for $(z_0, \cdots, z_n) \neq (0, \cdots, 0)$, which means that for $\lambda \in \mathbb{C}^*$, $[\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n]$ and $[z_0 : z_1 : \cdots : z_n]$ define the same point in $\mathbb{CP}^n$. The standard open covering of $\mathbb{CP}^n$ is given by

$$U_i = \{[z_0 : z_1 : \cdots : z_n] : z_i \neq 0\} \subset \mathbb{CP}^n. \tag{1.1.26}$$

It is open for the induced topology. Consider the bijective map $\varphi_i : U_i \to \mathbb{C}^n$ by

$$\varphi_i([z_0 : z_1 : \cdots : z_n]) = \left( \frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i} \right). \tag{1.1.27}$$

It is a homeomorphism. For the transition maps $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$, for

$$(\theta_1, \cdots, \theta_n) = \left( \frac{z_0}{z_j}, \cdots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \cdots, \frac{z_n}{z_j} \right) \in \mathbb{C}^n, \tag{1.1.28}$$

we may assume $i < j$ and get

$$\varphi_i \circ \varphi_j^{-1}(\theta_1, \cdots, \theta_n) = \left(\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i}\right)$$
$$= \left(\frac{\theta_1}{\theta_{i+1}}, \cdots, \frac{\theta_i}{\theta_{i+1}}, \frac{\theta_{i+2}}{\theta_{i+1}}, \cdots, \frac{\theta_j}{\theta_{i+1}}, \frac{1}{\theta_{i+1}}, \frac{\theta_{j+1}}{\theta_{i+1}} \cdots \frac{\theta_n}{\theta_{i+1}}\right). \quad (1.1.29)$$

These maps are obviously bijective and holomorphic.

Consider the $(1,1)$-form

$$\omega = \sqrt{-1}\partial\bar{\partial}\log(|z|^2) = \sqrt{-1} \cdot \frac{|z|^2\delta_{ij} - \bar{z}_i z_j}{|z|^4} dz^i \wedge d\bar{z}^j \quad (1.1.30)$$

on $\mathbb{C}^{n+1}\backslash\{0\}$. Observe that for $\lambda \in \mathbb{C}^*$,

$$\sqrt{-1}\partial\bar{\partial}\log(|\lambda z|^2) = \sqrt{-1}\partial\bar{\partial}(\log|\lambda|^2 + \log|z|^2)$$
$$= \sqrt{-1}\partial\bar{\partial}\log(|z|^2). \quad (1.1.31)$$

So from (1.1.25), the $(1,1)$-form in (1.1.30) induces a $(1,1)$-form $\omega_{\mathbb{CP}}$ on $\mathbb{CP}^n$. We claim that it is a Kähler form on $\mathbb{CP}^n$.

Since $\partial\bar{\partial}\log(|z_i|^2) = \partial\bar{\partial}(\log(z_i) + \log(\bar{z}_i)) = 0$, restricted on $U_i$, from (1.1.28), (1.1.30) and (1.1.31),

$$\omega_{\mathbb{CP}}|_{U_i} = \sqrt{-1}\partial\bar{\partial}\log(1+|\theta|^2) + \sqrt{-1}\partial\bar{\partial}\log(|z_j|^2) = \sqrt{-1}\partial\bar{\partial}\log(1+|\theta|^2)$$
$$= \sqrt{-1} \cdot \frac{(1+|\theta|^2)\delta_{kl} - \bar{\theta}_k\theta_l}{(1+|\theta|^2)^2} d\theta^k \wedge d\bar{\theta}^l. \quad (1.1.32)$$

Since the matrix $((1+|\theta|^2)\delta_{kl} - \bar{\theta}_k\theta_l)$ is positive definite, we obtain that $\omega_{\mathbb{CP}}$ is a Kähler form and $(\mathbb{CP}^n, \omega_{\mathbb{CP}})$ is a Kähler manifold. The metric induced by (1.1.15), which we denote by $g^{FS}$, is called the Fubini-Study metric. By (1.1.32), on $U_i$,

$$g^{FS}_{k\bar{l}} = \frac{\partial^2}{\partial\theta_k\partial\bar{\theta}_l}\log(1+|\theta|^2) = \frac{(1+|\theta|^2)\delta_{kl} - \bar{\theta}_k\theta_l}{(1+|\theta|^2)^2}. \quad (1.1.33)$$

In the followings, we will also denote the Kähler form $\omega_{\mathbb{CP}}$ by $\omega_{FS}$.

Remark that $\mathbb{CP}^n$ is simply connected. In fact, $\mathbb{CP}^n = S^{2n+1}/S^1$. From fibre exact sequence

$$\cdots \to \pi_1(S^{2n+1}) \to \pi_1(\mathbb{CP}^n) \to \pi_0(S^1) \to \cdots, \quad (1.1.34)$$

since $\pi_1(S^{2n+1}) = \pi_0(S^1) = \{1\}$, we have $\pi_1(\mathbb{CP}^n) = \{1\}$.